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On minimal $\mathcal{N} = 4$ topological strings and the $(1, k)$ minimal bosonic string

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Abstract: In this paper we consider tree-level scattering in the minimal $\mathcal{N} = 4$ topological string and show that a large class of N-point functions can be recast in terms of corresponding amplitudes in the $(1, k)$ minimal bosonic string. This suggests a non-trivial relation between the minimal $\mathcal{N} = 4$ topological strings, the $(1, k)$ minimal bosonic strings and their corresponding ADE matrix models. This relation has interesting and far-reaching implications for the topological sector of six-dimensional Little String Theories.

Keywords: Matrix Models, Superstrings and Heterotic Strings, Bosonic Strings, Topological Strings.

Contents

1. Introduction

The properties of topological strings have been studied over the years in much detail and by now it is fairly clear that topological strings are much more than a toy model for string theory (for a recent review see [1]). Indeed, many computations in topological string theory have been shown to have a direct consequence for ten-dimensional superstrings. For example, the $\mathcal{N} = 2$ topological string has played a role in the computation of low-energy superpotential terms in four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories and has led to an interesting connection between gauge theory effective superpotentials and zero-dimensional bosonic matrix models [2, 3]. Another application involves type II superstring theory compactified on a Calabi-Yau threefold. The four-dimensional low-energy effective action includes higher-derivative F-type terms of the form

$$\int d^4x d^4\theta (W_{ab}W^{ab})^g F_g(X^\Lambda) , \tag{1.1}$$

where W_{ab} is the graviphoton superfield of the $\mathcal{N} = 2$ supergravity theory and X^Λ are vector multiplets. The $\mathcal{N} = 2$ topological string computes the functions F_g [4, 5]. In recent developments, this fact has been used to formulate an intriguing relation between the topological string partition function and the partition function of dyonic BPS black holes in four-dimensional $\mathcal{N} = 2$ supergravity [6].

In the past, much work has been done on the $\mathcal{N} = 2$ topological string theory. In this case a worldsheet theory with $\mathcal{N} = (2, 2)$ supersymmetry is topologically twisted and the critical central charge is $\hat{c} = \frac{c}{3} = 3$. There are, however, other interesting examples where the amount of worldsheet supersymmetry and central charge are different and where a non-tirivial topological string theory can be defined. Most notably, worldsheet theories with $\mathcal{N} = (4, 4)$ supersymmetry and $\hat{c} = 2$ can be twisted appropriately to obtain the $\mathcal{N} = 4$ topological string. This theory has been argued [7] to be equivalent to the $\mathcal{N} = 2$ string,[1] which is obtained when the full $\mathcal{N} = 2$ superconformal group is gauged on the worldsheet [8 − 11]. In the same way that the $\mathcal{N} = 2$ topological string is relevant for type II string theory compactifications on Calabi-Yau threefolds, the $\mathcal{N} = 4$ topological string (or $\mathcal{N} = 2$ string) is directly relevant for type II string theory compactifications on K3 manifolds. The coefficient of certain terms of the form $R^4 F^{4g-4}$ in the low-energy effective action of the latter compactifications can be computed in the $\mathcal{N} = 4$ topological string [7]. The coefficient of a term of the form F^4, where F is an abelian gauge field in the low-energy theory, was the focus of the analysis of [12].

It is known [13 − 15] that the dynamics of string theory near an ADE singularity of a K3 compactification are captured by an asymptotically linear dilaton theory of the form

$$\mathbb{R}^{5,1} \times \left(\frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)} \right) / \mathbb{Z}_k \tag{1.2}$$

and that this background defines holographically a Little String Theory (LST) (for a review see [16, 17]). Via T-duality this background is also related to the near-horizon region of k parallel NS5-branes symmetrically arranged on a transverse circle. We can study the topological sector of these six-dimensional LST's by analyzing the $\mathcal{N} = 4$ topological string on $(\frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)})/\mathbb{Z}_k$. For obvious reasons, we will call this theory the minimal $\mathcal{N} = 4$ topological string. Earlier work on this theory appeared in [12, 18].

In this paper, we compute a large class of tree-level correlation functions in the above theory. In contrast to the $\mathcal{N} = 4$ topological string on $\mathbb{R}^4$ (or $\mathbb{R}^{2,2}$), which has vanishing N-point functions for $N \geq 4$ [9], we will see that the minimal $\mathcal{N} = 4$ topological string has non-vanishing amplitudes for any N. In fact, using recent results on tree-level correlation functions in $SL(2, \mathbb{R})$ [19 − 22] we can recast in many cases the topological string N-point function in terms of a corresponding N-point function in the $(1, k)$ minimal bosonic string. Among other things, this allows us to reformulate as a four-point function in the bosonic string, a four-point function that appeared in [12] and gives the coefficient of an F^4 term in the low-energy effective action of a K3 compactification near an ADE point of enhanced

[1] Not to be confused with the $\mathcal{N} = 2$ topological string.

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symmetry. We expect this observation to be particularly useful in completing the non-trivial check of heterotic-type II duality presented in [12] (related discussions on this F^4 term can be found in [23 – 25]).

The emergence of a bosonic string in the context of the minimal $\mathcal{N} = 4$ topological string reminds of the well-known equivalence between the $\mathcal{N} = 2$ topological string on $SL(2)_1/U(1)$ and the bosonic $c = 1$ string at self-dual radius [26 – 28].[2] In this paper we find evidence pointing towards a non-trivial relation between the minimal bosonic string and the minimal $\mathcal{N} = 4$ topological string.[3] It is natural to ask whether this relation is a true equivalence. This would have a non-trivial impact on the physics of six-dimensional LST's. For example, it would imply that the topological sector of these (yet quite mysterious) theories is completely solved by the ADE matrix models of ref. [33]. Some comments on this possible equivalence will be presented in the last section.

The organization of this paper is as follows. In section 2 we fix our notation and discuss the spectrum of the minimal $\mathcal{N} = 4$ topological string. In section 3 we compute a large class of N-point functions in the topological string and recast them in terms of corresponding amplitudes in the $(1, k)$ minimal bosonic string with the use of the Stoyanovsky-Ribault-Teschner (SRT) map. In section 4 we comment on the possibility of an equivalence between the minimal $\mathcal{N} = 4$ topological string and the $(1, k)$ minimal bosonic string and point out some implications. Four appendices contain useful facts and summarize our conventions.

Note added: while this paper was being prepared, we received the preprint [34], which has considerable overlap with this work. Our analysis deals with the same amplitudes and yields similar results, but is based on a slightly different treatment of the $\frac{SL(2)}{U(1)} \times \frac{SU(2)}{U(1)}$ CFT.

2. Minimal $\mathcal{N} = 4$ topological strings - the spectrum

2.1 Fixing the notation

The minimal $\mathcal{N} = 4$ topological string is defined by twisting topologically (in a manner that will be reviewed shortly) the $\mathcal{N} = 4$ superconformal field theory [35]

$$\left(\frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)}\right)/\mathbb{Z}_k \ . \tag{2.1}$$

The total central charge of this theory is

$$\hat{c}_{\text{tot}} = 1 + \frac{2}{k} + 1 - \frac{2}{k} = 2 \ . \tag{2.2}$$

As required, this is the critical value for the $\mathcal{N} = 4$ topological string. The $\mathbb{Z}_k$ orbifold in (2.1) restricts the spectrum to the states with integral $U(1)_R$ charge as needed by modular invariance. The conformal field theory (2.1) can be obtained from $SL(2)_k \times SU(2)_k$ with

[2] See also the recent discussions in [29 – 31].

[3] A relation between the $(1, k)$ minimal bosonic string and the A_k minimal $\mathcal{N} = 4$ topological string has been proposed also by Leonardo Rastelli in [32] based on work in collaboration with Martijn Wijnholt.

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a suitable $U(1)$ gauging. We summarize briefly a few of the relevant details to fix the notation.

The $\mathcal{N}=1$ superconformal field theory on $SU(2)_k$ is given by the bosonic $SU(2)_{k-2}$ WZW model with currents K^a, $a=1,2,3$ and three free Majorana fermions χ^a. The relevant OPE's are

$$K^3(z)K^3(w) \sim \frac{k-2}{2(z-w)^2}\ , \qquad K^+(z)K^-(w) \sim \frac{k-2}{(z-w)^2} + \frac{2K^3(w)}{z-w}\ , \tag{2.3}$$

$$K^3(z)K^\pm(w) \sim \pm\frac{K^\pm(w)}{z-w}\ , \tag{2.4}$$

$$\chi^+(z)\chi^-(w) \sim \frac{1}{z-w}\ , \qquad \chi^3(z)\chi^3(w) \sim \frac{1}{z-w}\ . \tag{2.5}$$

In terms of these currents the $\mathcal{N}=1$ superconformal generator takes the form

$$G^{(su)} = Q\Big(\frac{1}{\sqrt{2}}K^+\chi^- + \frac{1}{\sqrt{2}}K^-\chi^+ + K^3\chi^3 + \chi^+\chi^-\chi^3\Big)\ , \tag{2.6}$$

where $Q=\sqrt{\frac{2}{k}}$.

We will use a similar set of conventions for the supersymmetric $SL(2)_k$ WZW model. This theory consists of the bosonic $SL(2)_{k+2}$ WZW model with currents J^a $(a=1,2,3)$ and three free fermions ψ^a, which satisfy the OPE algebra

$$J^3(z)J^3(w) \sim -\frac{k+2}{2(z-w)^2}\ , \qquad J^+(z)J^-(w) \sim \frac{k+2}{(z-w)^2} - \frac{2J^3(w)}{z-w}\ , \tag{2.7}$$

$$J^3(z)J^\pm(w) \sim \pm\frac{J^\pm(w)}{z-w}\ , \tag{2.8}$$

$$\psi^+(z)\psi^-(w) \sim \frac{1}{z-w}\ , \qquad \psi^3(z)\psi^3(w) \sim -\frac{1}{z-w}\ . \tag{2.9}$$

The $SL(2)$ $\mathcal{N}=1$ supercurrent is

$$G^{(sl)} = Q\Big(\frac{1}{\sqrt{2}}J^+\psi^- + \frac{1}{\sqrt{2}}J^-\psi^+ - J^3\psi^3 - \psi^+\psi^-\psi^3\Big)\ . \tag{2.10}$$

The product CFT (2.1) can be obtained from the supersymmetric $SL(2)_k\times SU(2)_k$ theory by gauging the null $U(1)$ supercurrent (see [36, 37])

$$\frac{1}{\sqrt{2}}(\chi^3-\psi^3) - \theta\frac{Q}{\sqrt{2}}(J_3^{(tot)} - K_3^{(tot)})\ , \tag{2.11}$$

where by definition

$$J_3^{(tot)} = J_3 + \psi^+\psi^-\ ,\quad K_3^{(tot)} = K_3 + \chi^+\chi^-\ . \tag{2.12}$$

As usual, the gauging of the supercurrent (2.11) can be achieved by adding a bosonic (β',γ') ghost system with spin $\frac{1}{2}$, which contributes to the total BRST charge a term of the form

$$Q_{BRST} = \ldots + \int\frac{dz}{2\pi i}\gamma'(z)\frac{1}{\sqrt{2}}(\chi^3-\psi^3)\ . \tag{2.13}$$

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Then one can check up to BRST exact terms that the $\mathcal{N}=1$ supercurrent of the gauged theory is

$$G = G^{(su)} + G^{(sl)} = \frac{Q}{\sqrt{2}}(J^+\psi^- + J^-\psi^+ + K^+\chi^- + K^-\chi^+) \ . \tag{2.14}$$

It turns out that the gauged theory enjoys enhanced $\mathcal{N}=(4,4)$ worldsheet supersymmetry. The left-moving $\mathcal{N}=2$ superconformal generators take the form

$$\begin{aligned} G^\pm &= \frac{Q}{\sqrt{2}}\Big(e^{\mp iH_1}J^\pm + e^{\mp iH_2}K^\pm\Big) \ , \\ J &= -i\partial H_1 - i\partial H_2 \ , \end{aligned} \tag{2.15}$$

where H_1 and H_2 are two bosons that bosonize the free fermions $\psi^\pm$, $\chi^\pm$, *i.e.*

$$\psi^\pm = e^{\pm iH_1} \ , \ \ \chi^\pm = e^{\pm iH_2} \ . \tag{2.16}$$

For the right-moving sector we use the convention

$$\begin{aligned} \bar{G}^\pm &= \frac{Q}{\sqrt{2}}\Big(e^{\pm i\bar{H}_1}\bar{J}^\mp + e^{\pm i\bar{H}_2}\bar{K}^\mp\Big) \ , \\ \bar{J} &= i\bar{\partial} H_1 + i\bar{\partial} H_2 \ . \end{aligned} \tag{2.17}$$

The theory includes additional superconformal generators. They read

$$J^{\pm\pm} = e^{\pm\int J} = e^{\mp i(H_1+H_2)} \ , \qquad \bar{J}^{\pm\pm} = e^{\pm\int \bar{J}} = e^{\pm i(\bar{H}_1+\bar{H}_2)} \ , \tag{2.18}$$

$$\tilde{G}^\pm = \frac{Q}{\sqrt{2}}(e^{\mp iH_2}J^\mp + e^{\mp iH_1}K^\mp) \ , \qquad \bar{\tilde{G}}^\pm = \frac{Q}{\sqrt{2}}(e^{\pm i\bar{H}_2}\bar{J}^\pm + e^{\pm i\bar{H}_1}\bar{K}^\pm) \ . \tag{2.19}$$

One can check that the full set of currents $T(z), J(z), J^{\pm\pm}(z), G^\pm(z), \tilde{G}^\pm(z)$ satisfies the $\mathcal{N}=4$ superconformal algebra (see appendix A for more details on this algebra). The $\pm$ superscripts in (2.18), (2.19) denote the $U(1)_R$ charge of the corresponding generators. For example, the currents $J^{\pm\pm}$ have $U(1)_R$ charges ± 2 respectively and $\tilde{G}^\pm$ have $U(1)_R$ charges ± 1.

2.2 Physical states in $SL(2)/U(1) \times SU(2)/U(1)$

Physical states in $\frac{SL(2)}{U(1)} \times \frac{SU(2)}{U(1)}$ can be obtained by imposing the $U(1)$ gauging condition to a general vertex operator in $SL(2)\times SU(2)$. For example, consider the special set of vertex operators

$$\mathcal{V} = e^{i\sum_{i=1}^{2}(s_iH_i+\bar{s}_i\bar{H}_i)}\Phi^{(sl)}_{j_1,m_1,\bar{m}_1}\Phi^{(su)}_{j_2,m_2,\bar{m}_2} \ , \tag{2.20}$$

where $\Phi^{(sl)}_{j,m,\bar{m}}$, $\Phi^{(su)}_{j,m,\bar{m}}$ are respectively primary vertex operators of the bosonic $SL(2)_{k+2}$ and $SU(2)_{k-2}$ WZW models. The relevant conventions are summarized in appendix B. The null gauging condition $K_3^{(tot)} = J_3^{(tot)}$ implies the constraints

$$m_2 + s_2 = m_1 + s_1 \ , \ \ \bar{m}_2 + \bar{s}_2 = \bar{m}_1 + \bar{s}_1 \ . \tag{2.21}$$

For reference, we mention that the primary fields (2.20) have scaling dimension

$$\Delta = \frac{1}{2}(s_1^2+s_2^2) - \frac{j_1(j_1+1)}{k} + \frac{j_2(j_2+1)}{k} \tag{2.22}$$

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and $U(1)_R$ charge

$$q = -s_1 - s_2 \ . \tag{2.23}$$

In the context of the topological string a special role will be played by the NS sector chiral primary states. In particular, these states have the property $\Delta = \frac{q}{2}$ [38]. For the operators (2.20) this is the case when

$$\frac{1}{2}(s_1^2 + s_2^2) - \frac{j_1(j_1+1)}{k} + \frac{j_2(j_2+1)}{k} = -\frac{1}{2}(s_1 + s_2) \ . \tag{2.24}$$

For general level k and quantum numbers j (2.24) is satisfied if $j_1 = j_2$ or $-j_1 - 1 = j_2$ and

$$s_1^2 + s_2^2 = -s_1 - s_2 \ . \tag{2.25}$$

The last equation holds in the following four cases

$$(s_1, s_2) = \{(0,0), (0,-1), (-1,0), (-1,-1)\} \ . \tag{2.26}$$

Analogous statements apply to the right-moving sector and with obvious modifications to the anti-chiral states.

2.3 Physical states in the topological theory

A theory with $\mathcal{N} = 4$ superconformal symmetry can be twisted topologically in the same way as a theory with $\mathcal{N} = 2$ superconformal symmetry. Once we choose a $U(1)$ inside the $SU(2)$ R-symmetry group we can perform the familiar $\mathcal{N} = 2$ topological twist

$$T \to T + \frac{1}{2}\partial J \ , \quad \bar{T} \to \bar{T} \pm \frac{1}{2}\bar{\partial}\bar{J} \ . \tag{2.27}$$

As in the $\mathcal{N} = 2$ topological string, it is possible to make a choice of two inequivalent topological twists: the A-type corresponds to the minus sign in (2.27) and the B-type to the plus sign. Each of these choices is expected to be in one-to-one correspondence with an α- or β-type $\mathcal{N} = 2$ string [39]. For concreteness, in this paper we will consider the A-type topological twist.

The next and more crucial step is to decide how to define the BRST cohomology of the topologically twisted theory and how to couple the twisted theory to gravity. For the $\mathcal{N} = 2$ topological string it is enough to consider the cohomology of (c, c) or (c, a) fields ϕ_i. For the $\mathcal{N} = 4$ topological string the BRST cohomology is more constrained [7] and can be obtained by further restricting the fields ϕ_i to have the properties[4]

$$G^+\phi_i = 0 \ , \quad \tilde{G}^+\phi_i = 0 \ , \quad \bar{G}^-\phi_i = 0 \ , \quad \bar{\tilde{G}}^-\phi_i = 0 \tag{2.28}$$

and the equivalence relation

$$\phi_i \sim \phi_i + G^+\tilde{G}^+\bar{G}^-\bar{\tilde{G}}^-\chi \ . \tag{2.29}$$

[4] These relations define the appropriate BRST cohomology for the A-type topological twist. Similar relations define the BRST cohomology of the B-type topological string.

In addition, one has to impose the properties

$$
\begin{aligned}
J^{--}\phi_i &= M_i^{\bar{j}}\phi_{\bar{j}}^{\dagger}\ , \quad J^{++}\phi_i = 0\ , \\
\bar{J}^{++}\phi_i &= \bar{M}_i^{\bar{j}}\bar{\phi}_{\bar{j}}^{\dagger}\ , \quad \bar{J}^{--}\bar{\phi}_i = 0\ .
\end{aligned} \tag{2.30}
$$

The first and third equations in (2.30) impose the requirement that the fields $J^{--}\phi_i$ ($\bar{J}^{++}\phi_i$) can be expressed as a linear combination of antichiral (chiral) fields. $M_i^{\bar{j}}$ and $\bar{M}_i^{\tilde{j}}$ are the linear coefficients in this expansion.

For the vertex operators $\mathcal{V}$ in (2.20) the above requirements can be imposed in the following way (here we discuss the left-movers, but analogous statements apply also to the right-movers). The first condition in (2.28) is automatic for the chiral primary states. The second condition gives

$$
\begin{aligned}
&\int dz\ \tilde{G}^{+}(z)\cdot\mathcal{V}(0) = 0 \ \Leftrightarrow\ \int dz\ (e^{-iH_2}J^{-} + e^{-iH_1}K^{-})(z)\cdot\mathcal{V}(0) = 0 \Leftrightarrow \\
&\int dz\ \Big(z^{-s_2-1}(-j_1-1+m_1)e^{-iH_2(z)+is_2H_2(0)}\Phi^{(sl)}_{j_1,m_1-1,\bar{m}_1}(0)\cdots\Big) + \\
&\int dz\ \Big(z^{-s_1-1}(j_2+m_2)e^{-iH_1(z)+is_1H_1(0)}\Phi^{(su)}_{j_2,m_2-1,\bar{m}_2}(0)\cdots\Big) = 0\ .
\end{aligned} \tag{2.31}
$$

In this expression, the dots refer to the parts of the vertex operator $\mathcal{V}$, which are unaffected by the action of $\tilde{G}^{+}$. To derive (2.31) we made use of the OPE's

$$
J^{-}(z)\ \Phi^{(sl)}_{j,m}(w) \sim \frac{-j-1+m}{z-w}\ \Phi^{(sl)}_{j,m-1}(w)\ , \tag{2.32}
$$

$$
K^{-}(z)\ \Phi^{(su)}_{j,m}(w) \sim \frac{j+m}{z-w}\ \Phi^{(su)}_{j,m-1}(w)\ . \tag{2.33}
$$

The perhaps unorthodox factor $-j-1+m$ (instead of $j+m$) in (2.32) has its origin in our $SL(2)$ conventions, which are reviewed in appendix B. In a similar fashion, in order to check the first condition in (2.30) we need to compute

$$
\int dz\ J^{--}(z)\cdot\mathcal{V}(0) = \int dz\ z^{s_1+s_2}e^{i(H_1+H_2)(z)+i(s_1H_1+is_2H_2)(0)}\cdots\ , \tag{2.34}
$$

which should be expressible as a sum of antichiral primary fields. For the second condition in (2.30) we get

$$
\int dz\ J^{++}(z)\cdot\mathcal{V}(0) = \int dz\ z^{-s_1-s_2}e^{-i(H_1+H_2)(z)+i(s_1H_1+s_2H_2)(0)}\cdots = 0\ . \tag{2.35}
$$

Out of the four different cases in (2.26) the $(s_1,s_2)=(-1,-1)$ case fails to satisfy the condition $J^{--}\cdot\mathcal{V}_i = M_i^{\bar{j}}\mathcal{V}_j^{\dagger}$. Indeed, (2.34) gives

$$
J^{--}\cdot e^{-H_1-H_2}\Phi^{(sl)}_{j,m}\Phi^{(su)}_{j,m} = i(\partial H_1 + i\partial H_2)\Phi^{(sl)}_{j,m}\Phi^{(su)}_{j,m}\ , \tag{2.36}
$$

which is not an antichiral field. The remaining three cases $(s_1,s_2)=\{(0,0),(-1,0),(0,-1)\}$ satisfy the full set of conditions provided that we tune the quantum numbers j_i, m_i appropriately. We conclude that the BRST cohomology of the minimal $\mathcal{N}=4$ topological string

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contains (at least) the following states

$$\begin{aligned}\mathcal{O}_{2j+1} &= \Phi^{(sl)}_{-j-1,-j,-j}\Phi^{(su)}_{j,-j,-j}\ ,\\ \mathcal{O}^+_{2j+1} &= e^{-iH_1-i\bar{H}_1}\Phi^{(sl)}_{j,j+1,j+1}\Phi^{(su)}_{j,j,j}\ , \qquad \mathcal{O}^-_{2j+1} = e^{-iH_2-i\bar{H}_2}\Phi^{(sl)}_{j,-j-1,-j-1}\Phi^{(su)}_{j,-j,-j}\ .\end{aligned} \tag{2.37}$$

j is a half-integer with the property $0 \le j \le \frac{k-2}{2}$.

The vertex operators $\mathcal{O}^\pm_{2j+1}$ are normalizable in $\frac{SL(2)}{U(1)} \times \frac{SU(2)}{U(1)}$ (see appendix B for more details) and according to the general discussion in [12] their correlation functions compute amputated amplitudes in the holographic non-gravitational dual (see also [40]). These correlation functions will be computed in the next section. In addition, it will be useful to note that the vertex operators $\mathcal{O}^+_{2j+1}$ and $\mathcal{O}^-_{2j+1}$ are related via the reflection relation

$$\mathcal{O}^+_{2j+1} = \mathcal{O}^-_{k-2j-1}\ . \tag{2.38}$$

This is shown explicitly in appendix B.

In contrast, the zero R-charge vertex operators $\mathcal{O}_{2j+1}$ are non-normalizable. It is worth pointing out, however, that the normalizable version of $\mathcal{O}_{2j+1}$ is related to the $\mathcal{O}^-_{2j+1}$ state via the relation

$$\mathcal{O}^-_{2j+1} = \frac{k}{(2j+1)^2}\tilde{G}^+\bar{\tilde{G}}^- \cdot \left(\Phi^{(sl)}_{j,-j,-j}\Phi^{(su)}_{j,-j,-j}\right)\ . \tag{2.39}$$

2.4 Vertex operators in the $\mathcal{N} = 2$ string

Here we make a short parenthesis to discuss the corresponding vertex operators in the minimal $\mathcal{N} = 2$ string and to make contact with the analysis of [12]. Concentrating only on the left-moving sector, the physical $\mathcal{N} = 2$ string states $\hat{\mathcal{O}}^\pm_{2j+1}$ that appear in [12] take in our notation the following form[5]

$$\begin{aligned}\hat{\mathcal{O}}^+_{2j+1} &= e^{-\frac{1}{2}(\varphi_1+\varphi_2)+\frac{i}{2}(-H_1+H_2)}\Phi^{(sl)}_{j,j+1}\Phi^{(su)}_{j,j}\ ,\\ \hat{\mathcal{O}}^-_{2j+1} &= e^{-\frac{1}{2}(\varphi_1+\varphi_2)+\frac{i}{2}(H_1-H_2)}\Phi^{(sl)}_{j,-j-1}\Phi^{(su)}_{j,-j}\ .\end{aligned} \tag{2.40}$$

φ_1 and φ_2 are the two superconformal ghosts of the $\mathcal{N} = 2$ string associated with the generators G^- and G^+ respectively. The operators appearing in (2.40) are in the R sector. They can be converted to NS sector operators by using the spectral flow operation, a transformation that is gauged in the $\mathcal{N} = 2$ string. The operators that implement the spectral flow can be written as[6]

$$S^\pm = e^{\pm\frac{1}{2}(\varphi_2-\varphi_1)\mp\frac{i}{2}(H_1+H_2)}\ . \tag{2.41}$$

The basic properties of these operators are summarized in [12] (see pg. 44). Acting with them on the vertex operators $\hat{\mathcal{O}}^+_{2j+1}$ we find

$$\begin{aligned}S^-\hat{\mathcal{O}}^+_{2j+1} &= e^{-\varphi_2+iH_2}\Phi^{(sl)}_{j,j+1}\Phi^{(su)}_{j,j}\ , \qquad S^+\hat{\mathcal{O}}^+_{2j+1} = e^{-\varphi_1-iH_1}\Phi^{(sl)}_{j,j+1}\Phi^{(su)}_{j,j}\ ,\\ S^-\hat{\mathcal{O}}^-_{2j+1} &= e^{-\varphi_2+iH_1}\Phi^{(sl)}_{j,-j-1}\Phi^{(su)}_{j,-j}\ , \qquad S^+\hat{\mathcal{O}}^-_{2j+1} = e^{-\varphi_1-iH_2}\Phi^{(sl)}_{j,-j-1}\Phi^{(su)}_{j,-j}\ .\end{aligned} \tag{2.42}$$

[5] We drop an appropriate normalization factor. Also H, H' in [12] correspond to $-H_1$, $-H_2$ in our notation and $j_{there} = -j_{here} - 1$.

[6] As in [12] we omit a factor $e^{\pm\frac{1}{2}c\tilde{b}}$.

In the $\mathcal{N}=2$ string there are two picture changing operators[7]

$$Z^- = e^{\varphi_1}[G^- + ...] \ , \quad Z^+ = e^{\varphi_2}[G^+ + ...] \tag{2.43}$$

and we can use them to change the picture of the vertex operators (2.42) as follows

$$\begin{aligned} S^+ \hat{\mathcal{O}}^+_{2j+1} &\sim -\frac{\sqrt{2}}{Q(2j+1)} e^{-\varphi_1-\varphi_2} \Phi^{(sl)}_{j,j} \Phi^{(su)}_{j,j} \ , \\ S^- \hat{\mathcal{O}}^-_{2j+1} &\sim -\frac{\sqrt{2}}{Q(2j+1)} e^{-\varphi_1-\varphi_2} \Phi^{(sl)}_{j,-j} \Phi^{(su)}_{j,-j} \ . \end{aligned} \tag{2.44}$$

We made use of the identities

$$\begin{aligned} G^+ \cdot \left(\Phi^{(sl)}_{j,j} \Phi^{(su)}_{j,j}\right) &= -\frac{Q}{\sqrt{2}}(2j+1) e^{-iH_1} \Phi^{(sl)}_{j,j+1} \Phi^{(su)}_{j,j} \ , \\ G^- \cdot \left(\Phi^{(sl)}_{j,-j} \Phi^{(su)}_{j,-j}\right) &= -\frac{Q}{\sqrt{2}}(2j+1) e^{iH_1} \Phi^{(sl)}_{j,-j-1} \Phi^{(su)}_{j,-j} \ . \end{aligned} \tag{2.45}$$

As explained in [7], vertex operators in the $\mathcal{N}=2$ string of the form

$$\hat{\mathcal{O}} = c \ e^{-\varphi_1-\varphi_2} \ V \tag{2.46}$$

correspond in the $\mathcal{N}=4$ topological string to the $\tilde{G}^+$-exact vertex operators

$$\mathcal{O} = \tilde{G}^+ V \ . \tag{2.47}$$

Combining eqs. (2.37), (2.39) and (2.44) we deduce the correspondence

$$\hat{\mathcal{O}}^-_{2j+1} \leftrightarrow \mathcal{O}^-_{2j+1} \ . \tag{2.48}$$

The additional correspondence

$$\hat{\mathcal{O}}^+_{2j+1} \leftrightarrow \mathcal{O}^+_{2j+1} \tag{2.49}$$

follows from the $\mathcal{N}=4$ topological string reflection relation (2.38) and the analogous relation in the $\mathcal{N}=2$ string $\hat{\mathcal{O}}^+_{2j+1} = \hat{\mathcal{O}}^-_{k-2j-1}$ (see [12, eq. (5.34)]).

3. Minimal $\mathcal{N}=4$ topological strings - N-point functions

In this section we compute the tree-level correlation functions of the vertex operators $\mathcal{O}^{\pm}_{2j+1}$. First, we consider the three- and four-point functions and then proceed with a general computation of N-point functions for an arbitrary number of insertions $N \geq 5$. Using the Stoyanovsky-Ribault-Teschner map we show how to recast these correlation functions in terms of corresponding amplitudes in the minimal *bosonic* string.

[7] The ellipses denote a set of terms that depend only on ghosts.

3.1 3-point functions

In the $\mathcal{N}=4$ topological string the three-point functions of $\tilde{G}^+\bar{\tilde{G}}^-$-exact operators $\phi_i = \tilde{G}^+\bar{\tilde{G}}^- V_i$ take the form [7]

$$c_{ijk} = \langle \phi_i \phi_j V_k \rangle \ . \tag{3.1}$$

With this definition the three-point function is symmetric in the labels (i,j,k) and respects all the invariances of the $\mathcal{N}=4$ topological string. In this section, we will focus on correlation functions of the vertex operators $\mathcal{O}^-_{2j+1}$. A general amplitude with both $\mathcal{O}^+_{2j+1}$ and $\mathcal{O}^-_{2j+1}$ insertions follows trivially by using the reflection relation (2.38).

For reasons that will become more apparent in the following, a vertex operator $\mathcal{O}^-_{2j+1}$ will be inserted in the amplitude in the $U(1)$ gauge equivalent form[8]

$$\mathcal{O}^-_{2j+1} \sim e^{\int (J_g + \bar{J}_g)} \mathcal{O}^-_{2j+1} = e^{-iH_1 - i\bar{H}_1} \Phi^{(sl),w=1}_{j,-j-1,-j-1} \Phi^{(su),w=1}_{j,-j,-j} \ . \tag{3.2}$$

Recall that we gauge the null $U(1)$ current

$$J_g = i\partial H_2 - i\partial H_1 + K_3 - J_3 \ . \tag{3.3}$$

This current commutes with all the generators of the $\mathcal{N}=4$ superconformal algebra. When it acts on $\mathcal{O}^-_{2j+1}$ as in (3.2), part of the effect is to spectral flow the $SL(2)$ and $SU(2)$ components of the vertex operator. More details on our spectral flow conventions can be found in appendix B. Moreover, the uncharged vertex operator V_{2j+1} takes for $\mathcal{O}^-_{2j+1}$ the form (see eq. (2.39))

$$V_{2j+1} = \frac{k}{(2j+1)^2} \Phi^{(sl)}_{j,-j,-j} \Phi^{(su)}_{j,-j,-j} \ . \tag{3.4}$$

As in the $\mathcal{N}=2$ topological string theory [41, 42], a correlation function in the $\mathcal{N}=4$ topological string can be computed in the NS sector of the untwisted theory by inserting an extra spectral flow operator with $\theta = 1$ to saturate the background charge $-\hat{c} = -2$ induced by the topological twist. In our case, the appropriate insertion is the $\mathcal{N}=4$ current J^{--}. For instance, the 3-point function (3.1) can be computed in the untwisted theory as

$$c_{j_1 j_2 j_3} = \left\langle \left(J'^{--} \bar{J}'^{--} \mathcal{O}^-_{2j_1+1} \right) \mathcal{O}^-_{2j_2+1} V_{2j_3+1} \right\rangle \ , \tag{3.5}$$

where we have defined the $U(1)$ gauge equivalent current J'^{--} as

$$J'^{--} = e^{-\int (J + J_g)} = J^{--} e^{-\int J_g} \ . \tag{3.6}$$

As a result,

$$J'^{--} \bar{J}'^{--} \mathcal{O}^-_{2j_1+1} = e^{iH_1 + i\bar{H}_1} \Phi^{(sl)}_{j,-j-1,-j-1} \Phi^{(su)}_{j,-j,-j} \tag{3.7}$$

[8] A similar trick was also used in the $\mathcal{N}=2$ topological string [29, 30].

and we see that the H_1, $\bar{H}_1$ momentum in (3.5) is automatically conserved. Thus, (3.5) reduces to[9]

$$c_{j_1 j_2 j_3} = \frac{k}{(2j_3+1)^2}\Big\langle \Phi^{(sl)}_{j_1,-j_1-1,-j_1-1}\Phi^{(sl),w=1}_{j_2,-j_2-1,-j_2-1}\Phi^{(sl)}_{j_3,-j_3,-j_3}\Big\rangle \times \Big\langle \Phi^{(su)}_{j_1,-j_1,-j_1}\Phi^{(su),w=1}_{j_2,-j_2,-j_2}\Phi^{(su)}_{j_3,-j_3,-j_3}\Big\rangle . \tag{3.8}$$

This amplitude can be computed explicitly using known results for the three-point functions in the $SL(2)$ and $SU(2)$ WZW models [43–45]. Since this is essentially the computation performed in [12] we will not repeat it here. Instead, we proceed to calculate (3.8) with the use of the SRT map [20, 22], which is summarized in appendix C. The fact that the SRT map can be very useful in the context of the topological strings was pointed out for the first time in [30]. For the $SL(2)$ part in (3.8) we get a maximally winding number violating amplitude, which can be recast into the form

$$\Big\langle \Phi^{(sl)}_{j_1,-j_1-1,-j_1-1}\Phi^{(sl),w=1}_{j_2,-j_2-1,-j_2-1}\Phi^{(sl)}_{j_3,-j_3,-j_3}\Big\rangle = -\delta\Big(j_1+j_2+j_3-\frac{k-2}{2}\Big) \frac{2\pi^{-3}}{\sqrt{k}}c_{k+2}(2j_3+1)^2\prod_{i=1}^{3}\gamma(-1-2j_i)\Big\langle \mathcal{V}_{\frac{j_1+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\mathcal{V}_{\frac{j_2+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\mathcal{V}_{\frac{j_3+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\Big\rangle_{Liouville} . \tag{3.9}$$

We are using the standard γ-function notation

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} . \tag{3.10}$$

The three-point function on the r.h.s. is a three-point function in Liouville theory with linear dilaton slope $Q = \frac{1}{\sqrt{k}} + \sqrt{k}$. The vertex operators $\mathcal{V}_a$ are primary fields of this theory with asymptotic (weak-coupling) form

$$\mathcal{V}_a = e^{\sqrt{2}a\phi} \tag{3.11}$$

and scaling dimension $\Delta(\mathcal{V}_a) = a(Q-a)$. ϕ is the Liouville field. In (3.9) c_{k+2} is a (possibly diverging) k-dependent constant, which is part of the SRT map [22].

With analytic continuation in k (*i.e.* $k \to -k$) we obtain similar expressions for the $SU(2)$ part

$$\Big\langle \Phi^{(su)}_{j_1,-j_1,-j_1}\Phi^{(su),w=1}_{j_2,-j_2,-j_2}\Phi^{(su)}_{j_3,-j_3,-j_3}\Big\rangle = -\delta\Big(j_1+j_2+j_3-\frac{k-2}{2}\Big) i\frac{2\pi^{-3}}{\sqrt{k}}\tilde{c}_{k-2}\prod_{i=1}^{3}\gamma(1+2j_i)\Big\langle \widetilde{\mathcal{V}}_{\frac{j_1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\widetilde{\mathcal{V}}_{\frac{j_2}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\widetilde{\mathcal{V}}_{\frac{j_3}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\Big\rangle_{Coulomb-Gas} . \tag{3.12}$$

The three-point function on the r.h.s. of this equation is a three-point function in the Coulomb-Gas representation of the $(1,k)$ minimal model. In this representation the vertex operators $\tilde{\mathcal{V}}_a$ are

$$\tilde{\mathcal{V}}_a = e^{i\sqrt{2}a\tilde{\phi}} , \tag{3.13}$$

[9] In this paper we omit certain extra factors that depend explicitly on the worldsheet variables. When taken into account properly they cancel out at the end of the computation and isolate the dimensionless part of the untwisted N-point functions. See [30] for a careful treatment of a similar calculation in the $\mathcal{N}=2$ topological string.

where $\tilde{\phi}$ is the boson of the Coulomb-Gas representation. The scaling dimension of $\tilde{\mathcal{V}}_a$ is $\Delta(\tilde{\mathcal{V}}_a) = a(\tilde{Q}+a)$, where $\tilde{Q} = \frac{1}{\sqrt{k}} - \sqrt{k}$.

Combining the above expressions with the fact that[10]

$$\mathcal{T}_{2j+1} = c\bar{c}\; \mathcal{V}_{\frac{j+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}} \widetilde{\mathcal{V}}_{\frac{j}{\sqrt{k}}+\frac{\sqrt{k}}{2}} \tag{3.14}$$

is a physical tachyon vertex operator in the $(1,k)$ minimal bosonic string we obtain

$$c_{j_1j_2j_3} = -i\delta\left(j_1+j_2+j_3-\frac{k-2}{2}\right)4\pi^{-6}c_{k+2}\tilde{c}_{k-2}\prod_{i=1}^{3}\frac{1}{(1+2j_i)^2}\left\langle \mathcal{T}_{2j_1+1}\mathcal{T}_{2j_2+1}\mathcal{T}_{2j_3+1}\right\rangle_{(1,k)} . \tag{3.15}$$

We see that, up to a normalization factor, the $\mathcal{N}=4$ topological string three-point function $c_{j_1j_2j_3}$ is the same as a corresponding three-point function in the $(1,k)$ minimal bosonic string. In addition, (3.15) suggests a correspondence between the $\mathcal{N}=4$ topological string vertex operator $\mathcal{O}^-_{2j+1}$ and the $(1,k)$ minimal bosonic string tachyon $\mathcal{T}_{2j+1}$. Later in this section, we will test this identification by computing higher N-point functions.

3.2 4-point functions

It is known [7] that the general four-point function in the $\mathcal{N}=4$ topological string takes the form

$$c_{j_1j_2j_3j_4} = \left\langle \phi_{j_1}\phi_{j_2}\left(J^{--}\bar{J}^{--}\phi_{j_3}\right)\int d^2z_4\phi_{j_4}\right\rangle . \tag{3.16}$$

As in the previous subsection, here also we will focus on the correlation functions of the vertex operators $\mathcal{O}^-_{2j+1}$. Again, we can compute this amplitude in the untwisted theory by inserting an extra J^{--} current to saturate the background charge induced by the topological twist. The untwisted theory amplitude reads

$$c_{j_1j_2j_3j_4} = \left\langle\left(J'^{--}\bar{J}'^{--}\mathcal{O}^-_{2j_1+1}\right)\mathcal{O}^-_{2j_2+1}\left(J'^{--}\bar{J}'^{--}\mathcal{O}^-_{2j_3+1}\right)\int d^2z_4\;\mathcal{O}^-_{2j_4+1}\right\rangle . \tag{3.17}$$

Using the expressions (3.2) and (3.7) we see immediately that the H_1, $\bar{H}_1$ momentum in (3.17) is automatically conserved and that the amplitude reduces to a product of $SL(2)$ and $SU(2)$ parts

$$\begin{aligned} c_{j_1j_2j_3j_4} = & \int d^2z_4\left\langle \Phi^{(sl)}_{j_1,-j_1-1,-j_1-1}\Phi^{(sl),w=1}_{j_2,-j_2-1,-j_2-1}\Phi^{(sl)}_{j_3,-j_3-1,-j_3-1}\Phi^{(sl),w=1}_{j_4,-j_4-1,-j_4-1}\right\rangle \times \\ & \left\langle \Phi^{(su)}_{j_1,-j_1,-j_1}\Phi^{(su),w=1}_{j_2,-j_2,-j_2}\Phi^{(su)}_{j_3,-j_3,-j_3}\Phi^{(su),w=1}_{j_4,-j_4,-j_4}\right\rangle . \end{aligned} \tag{3.18}$$

With the use of the SRT map we obtain the following results. For the $SL(2)$ and $SU(2)$ parts respectively we find

$$\left\langle \Phi^{(sl)}_{j_1,-j_1-1,-j_1-1}\Phi^{(sl),w=1}_{j_2,-j_2-1,-j_2-1}\Phi^{(sl)}_{j_3,-j_3-1,-j_3-1}\Phi^{(sl),w=1}_{j_4,-j_4-1,-j_4-1}\right\rangle = \frac{2c^2_{k+2}}{\pi^5\sqrt{k}} \times \tag{3.19}$$

[10] For a summary of the basic properties of the $(1,k)$ minimal bosonic string see appendix D.

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$$\delta\Big(\sum_{i=1}^{4} j_i-(k-2)\Big)\prod_{i=1}^{4}\gamma(-1-2j_i)\Big\langle\mathcal{V}_{\frac{j_1+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\mathcal{V}_{\frac{j_2+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\mathcal{V}_{\frac{j_3+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\mathcal{V}_{\frac{j_4+1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\Big\rangle_{Liouville},$$

$$\Big\langle\Phi^{(su)}_{j_1,-j_1,-j_1}\Phi^{(su),w=1}_{j_2,-j_2,-j_2}\Phi^{(su)}_{j_3,-j_3,-j_3}\Phi^{(su),w=1}_{j_4,-j_4,-j_4}\Big\rangle=-\frac{2i\ \tilde{c}^2_{k-2}}{\pi^5\sqrt{k}}\prod_{i=1}^{4}\gamma(1+2j_i)\times$$
$$\delta\Big(\sum_{i=1}^{4} j_i-(k-2)\Big)\Big\langle\tilde{\mathcal{V}}_{\frac{j_1}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\tilde{\mathcal{V}}_{\frac{j_2}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\tilde{\mathcal{V}}_{\frac{j_3}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\tilde{\mathcal{V}}_{\frac{j_4}{\sqrt{k}}+\frac{\sqrt{k}}{2}}\Big\rangle_{Coulomb-Gas}. \tag{3.20}$$

From (3.18), (3.19) and (3.20) we deduce that the topological string four-point function $c_{j_1j_2j_3j_4}$ can be written as

$$c_{j_1j_2j_3j_4}=-\frac{4i\ c^2_{k+2}\tilde{c}^2_{k-2}}{\pi^{10}k}\delta\Big(\sum_{i=1}^{4} j_i-(k-2)\Big)\prod_{i=1}^{4}\frac{1}{(1+2j_i)^2}\times$$
$$\Big\langle\mathcal{T}_{2j_1+1}\mathcal{T}_{2j_2+1}\mathcal{T}_{2j_3+1}\int d^2z_4\ \mathcal{T}_{2j_4+1}\Big\rangle_{(1,k)}. \tag{3.21}$$

This is the four-point function generalization of eq. (3.15) above. As before, we find that, up to a normalization factor that can be absorbed into the definition of the $\mathcal{N}=4$ topological string vertex operators, the four-point functions $c_{j_1j_2j_3j_4}$ are captured by corresponding four-point functions in the $(1,k)$ minimal bosonic string.

In the minimal bosonic string it is known independently [46] (see also [47]) that

$$\delta\Big(\sum_{i=1}^{4} j_i-(k-1)\Big)\Big\langle\mathcal{T}_{2j_1+1}\mathcal{T}_{2j_2+1}\mathcal{T}_{2j_3+1}\int d^2z_4\ \mathcal{T}_{2j_4+1}\Big\rangle_{(1,k)}=$$
$$\delta\Big(\sum_{i=1}^{4} j_i-(k-1)\Big)\min(2j_i,k-2j_i-1)\ . \tag{3.22}$$

Unfortunately, we cannot make use of this result to determine the amplitude appearing in the r.h.s. of (3.21), because the selection rule is different. On the other hand, from [12, eq. (6.8)] we expect the $\mathcal{N}=4$ topological string amplitude

$$c_{j_1j_2j_3j_4}=\delta\Big(\sum_{i=1}^{4} j_i-(k-2)\Big)\min(2j_i+1,k-2j_i-1)\ , \tag{3.23}$$

where an unimportant normalization factor has been dropped. This result was anticipated in [12] as a non-trivial consistency requirement for heterotic-type II string duality. Here we find it to be equivalent (up to a normalization factor) to the following statement in the minimal bosonic string

$$\delta\Big(\sum_{i=1}^{4} j_i-(k-2)\Big)\Big\langle\mathcal{T}_{2j_1+1}\mathcal{T}_{2j_2+1}\mathcal{T}_{2j_3+1}\int d^2z_4\ \mathcal{T}_{2j_4+1}\Big\rangle_{(1,k)}=$$
$$\delta\Big(\sum_{i=1}^{4} j_i-(k-2)\Big)\min(2j_i+1,k-2j_i-1)\ . \tag{3.24}$$

We leave an explicit derivation of this result to a future publication. Moreover, it would be interesting to obtain a better understanding of the rôle of the above selection rules in the emerging relation between the minimal bosonic string and the minimal $\mathcal{N}=4$ topological string.

3.3 N-point functions

For $N \geq 5$ the N-point amplitudes of the $\mathcal{N}=4$ topological string take the form (*c.f.* [7])

$$c_{j_1 j_2 \dots j_N} = \Big\langle \Big(J'^{--} \bar{J}'^{--} \mathcal{O}^{-}_{2j_1+1} \Big) \mathcal{O}^{-}_{2j_2+1} \Big(J'^{--} \bar{J}'^{--} \mathcal{O}^{-}_{2j_3+1} \Big) \times \int d^2 z_4 \; \mathcal{O}^{-}_{2j_4+1} \prod_{\ell=5}^{N} \int d^2 z_\ell \; \widehat{G}^{-} \widehat{\bar{G}}^{+} \cdot \mathcal{O}^{-}_{2j_\ell+1} \Big\rangle \; . \tag{3.25}$$

The generators $\widehat{G}^{-}$, $\widehat{\bar{G}}^{+}$ are defined as

$$\widehat{G}^{-} = u_1 G^{-} + u_2 \tilde{G}^{-} \; , \quad \widehat{\bar{G}}^{+} = \bar{u}_1 \bar{G}^{+} + \bar{u}_2 \tilde{\bar{G}}^{+} \; , \tag{3.26}$$

where u_1, u_2 and $\bar{u}_1, \bar{u}_2$ are two independent sets of complex numbers with the property

$$|u_1|^2 + |u_2|^2 = 1 \; , \quad |\bar{u}_1|^2 + |\bar{u}_2|^2 = 1 \; . \tag{3.27}$$

As a result, the amplitude (3.25) is a homogeneous polynomial of the parameters $u_1, u_2, \bar{u}_1$, $\bar{u}_2$ of degree $2(N-4)$. The polynomial expansion can be written as

$$c_{j_1 j_2 \dots j_N}(u_1, u_2; \bar{u}_1, \bar{u}_2) = \sum_{n=0}^{N-4} \mathcal{C}^{n}_{j_1 j_2 \dots j_N} \; (u_1 \bar{u}_1)^n (u_2 \bar{u}_2)^{N-4-n} \; , \tag{3.28}$$

where $\mathcal{C}^{n}_{j_1 j_2 \dots j_N}$ are non-trivial data of the $\mathcal{N}=4$ topological string that can be computed by calculating (3.25).

This can be achieved by applying the SRT map as before. First we need to determine the explicit form of $\widehat{G}^{-} \widehat{\bar{G}}^{+} \cdot \mathcal{O}^{-}_{2j+1}$. By straightforward algebra and the use of the identity $\Phi^{(su),w=1}_{j,-j} = \Phi^{(su)}_{\frac{k-2}{2}-j,\frac{k-2}{2}-j}$ we find

$$G^{-} \cdot \mathcal{O}^{-}_{2j+1} = \frac{1}{\sqrt{k}} \Big[(-2j-2) \Phi^{(sl),w=1}_{j,-j-2} \Phi^{(su),w=1}_{j,-j} + (k-2-2j) e^{-iH_1 + iH_2} \Phi^{(sl),w=1}_{j,-j-1} \Phi^{(su)}_{\frac{k-2}{2}-j,\frac{k-2}{2}-j-1} \Big] \; . \tag{3.29}$$

In the BRST cohomology of the coset, this is equivalent to

$$G^{-} \cdot \mathcal{O}^{-}_{2j+1} = \frac{1}{\sqrt{k}} \Big[(-2j-2) \Phi^{(sl),w=1}_{j,-j-2} \Phi^{(su),w=1}_{j,-j} + (k-2-2j) \Phi^{(sl)}_{j,-j-1} \Phi^{(su),w=-1}_{\frac{k-2}{2}-j,\frac{k-2}{2}-j-1} \Big] \; . \tag{3.30}$$

In a similar fashion,

$$\tilde{G}^{-} \cdot \mathcal{O}^{-}_{2j+1} = \frac{1}{\sqrt{k}} \Big[(k-2j) \; \Phi^{(sl),w=-1}_{-j+\frac{k-2}{2},-j+\frac{k}{2}+1} \Phi^{(su)}_{j,-j} + 2j \; \Phi^{(sl),w=1}_{j,-j-1} \Phi^{(su),w=1}_{j,-j+1} \Big] \; . \tag{3.31}$$

The action of $\widehat{\bar{G}}^{+}$ on the right-movers is identical.

It is clear from the above expressions that the general coefficient $\mathcal{C}^n_{j_1 j_2 \cdots j_N}$ will not reduce to a product of $SL(2)$ and $SU(2)$ amplitudes with maximal winding number violation. In fact, the winding number violation and the associated selection rules will depend crucially on how many times we insert the second term of (3.30) or the first term of (3.31) into the amplitude (3.25).[11] In order to illustrate the computation of the general coefficient $\mathcal{C}^n_{j_1 j_2 \cdots j_N}$ let us consider in detail a few representative cases.

Consider, for example, the case of the amplitude $\mathcal{C}^{N-4}_{j_1 j_2 \cdots j_N}$. It takes the form

$$\begin{aligned}\mathcal{C}^{N-4}_{j_1 j_2 \cdots j_N} &= \Big\langle \Big(J'^{--} \bar{J}'^{--} \mathcal{O}^-_{2j_1+1} \Big) \mathcal{O}^-_{2j_2+1} \Big(J'^{--} \bar{J}'^{--} \mathcal{O}^-_{2j_3+1} \Big) \times \\ & \quad \int d^2 z_4 \; \mathcal{O}^-_{2j_4+1} \prod_{\ell=5}^{N} \int d^2 z_\ell \; G^- \bar{G}^+ \cdot \mathcal{O}^-_{2j_\ell+1} \Big\rangle \, . \end{aligned} \tag{3.32}$$

We can choose the selection rule appropriately, so that (3.32) does not receive any contributions from insertions of $G^{(su)-}$ (see the second term in (3.30)). In that case, (3.32) reduces to the following product of maximally winding number violating $SL(2)$ and $SU(2)$ amplitudes

$$\begin{aligned}\mathcal{C}^{N-4}_{j_1 j_2 \cdots j_N} &= \frac{1}{k^{N-4}} \int d^2 z_4 \prod_{\ell=4}^{N} \int d^2 z_\ell \; (2j_\ell + 2)^2 \\ & \Big\langle \Phi^{(sl)}_{j_1,-j_1-1,-j_1-1} \Phi^{(sl),w=1}_{j_2,-j_2-1,-j_2-1} \Phi^{(sl)}_{j_3,-j_3-1,-j_3-1} \Phi^{(sl),w=1}_{j_4,-j_4-1,-j_4-1} \Phi^{(sl),w=1}_{j_\ell,-j_\ell-2,-j_\ell-2} \Big\rangle \times \\ & \Big\langle \Phi^{(su)}_{j_1,-j_1,-j_1} \Phi^{(su),w=1}_{j_2,-j_2,-j_2} \Phi^{(su)}_{j_3,-j_3,-j_3} \Phi^{(su),w=1}_{j_4,-j_4,-j_4} \Phi^{(su),w=1}_{j_\ell,-j_\ell,-j_\ell} \Big\rangle \, . \end{aligned} \tag{3.33}$$

The H_1 and $\bar{H}_1$ momenta are automatically conserved as always. Then by direct application of the SRT map we find again that we can recast the $\mathcal{N}=4$ topological string amplitude $\mathcal{C}^{N-4}_{j_1 j_2 \cdots j_N}$ in terms of an amplitude in the $(1,k)$ minimal bosonic string

$$\begin{aligned}\mathcal{C}^{N-4}_{j_1 j_2 \cdots j_N} &= \frac{-4i\pi^{6-4N}}{k^{N-3}} (c_{k+2}\tilde{c}_{k-2})^{N-2} \prod_{i=1}^{N} \frac{1}{(1+2j_i)^2} \delta\Big(\sum_{i=1}^{N} j_i - \frac{k-2}{2}(N-2) \Big) \times \\ & \quad \Big\langle \mathcal{T}_{2j_1+1} \mathcal{T}_{2j_2+1} \mathcal{T}_{2j_3+1} \prod_{\ell=4}^{N} \int d^2 z_\ell \; \mathcal{T}_{2j_\ell+1} \Big\rangle_{(1,k)} \, . \end{aligned} \tag{3.34}$$

This computation generalizes to all the coefficients $\mathcal{C}^n_{j_1 j_2 \cdots j_N}$ (for $n = 0, ..., N-4$) without any $e^{iH_2} K^-$ or $e^{iH_2} J^+$ insertions. In that case, the general result is

$$\begin{aligned}\mathcal{C}^{n}_{j_1 j_2 \cdots j_N} &= \frac{-4i\pi^{6-4N}}{k^{N-3}} (c_{k+2}\tilde{c}_{k-2})^{N-2} \prod_{i=1}^{N} \frac{1}{(1+2j_i)^2} \delta\Big(\sum_{i=1}^{N} j_i - \frac{k-2}{2}(N-2) - n \Big) \times \\ & \quad \Big\langle \mathcal{T}_{2j_1+1} \mathcal{T}_{2j_2+1} \mathcal{T}_{2j_3+1} \prod_{\ell=4}^{N} \int d^2 z_\ell \; \mathcal{T}_{2j_\ell+1} \Big\rangle_{(1,k)} \, . \end{aligned} \tag{3.35}$$

[11] I would like to thank D. Sahakyan and T. Takayanagi for a useful discussion on this point.

There are, however, many other cases, with a non-zero number of $e^{iH_2}K^-$ or $e^{iH_2}J^+$ insertions that have different selection rules and do not reduce to maximally winding number violating amplitudes. Consider, for example, the amplitude (3.32) with selection rules $\sum_{i=1}^N j_i = \frac{k-2}{2}(N-2-\ell)-\ell$ for $\ell = 1,2,...,N-4$. In this case, (3.32) will receive contributions from those terms that have $N-4-\ell$ $G^{(sl)-} = e^{iH_1}J^-$ insertions and ℓ $G^{(su)-} = e^{iH_2}K^-$ insertions. Since they do not exhibit maximal winding number violation these correlation functions do not appear to have a clear interpretation in the minimal bosonic string. After the application of the SRT map the amplitudes will involve extra integrated insertions of degenerate vertex operators. In particular, this makes the identification of vertex operators $\mathcal{O}^-_{2j+1} \leftrightarrow \mathcal{T}_{2j+1}$ less straightforward. Similar statements apply to $\mathcal{C}^n_{j_1 j_2 \cdots j_N}$ for any $n = 0,1,...,N-4$. It is important to understand this point further.

4. A web of equivalences

In this paper we identified a certain class of physical states in the minimal $\mathcal{N}=4$ topological string and proceeded to analyze their N-point functions for arbitrary N. With the use of the SRT map, we found that we can recast a subset of these correlation functions in terms of corresponding amplitudes in the $(1,k)$ minimal bosonic string. The results presented in this paper have immediate consequences for the topological sector of little string theory. With the use of the minimal bosonic string we can, in principle, deduce in little string theory the precise value of a large number of correlation functions of off-shell observables. For example, we can recast a result anticipated in [12] as a non-trivial requirement of heterotic-type II string duality (see eq. (3.23)) as a statement in the minimal bosonic string, which we expect can be verified explicitly. It would be interesting to explore this point further with an explicit computation of the minimal bosonic string amplitudes and see how much information we can obtain about the structure of little string theory from the correlation functions presented in this paper. We hope to return to this issue in a future publication.

More generally, the above results suggest an interesting relation between the $\mathcal{N}=4$ topological string and the $(1,k)$ minimal bosonic string.[12] It is natural to ask whether this relation is a full-fledged equivalence like the equivalence between the $\mathcal{N}=2$ topological string on $SL(2)_1/U(1)$ and the $c=1$ non-critical bosonic string at self-dual radius [26]. For this it is crucial to obtain a better understanding of the amplitudes (3.25) with general selection rules and to examine whether a similar relation persists for amplitudes that involve generic states in the cohomology of the minimal $\mathcal{N}=4$ topological string. The latter requires a full classification of the observables of the minimal $\mathcal{N}=4$ topological string, which is beyond the immediate scope of this note.

Clearly, a full equivalence would be a powerful statement with many interesting implications. For instance, when combined with other known facts, it would imply that the $(1,k)$ minimal bosonic string is equivalent to the following seemingly unrelated theories:

(1) The minimal $\mathcal{N}=4$ topological string, which is also related via T-duality to the topological string on K3 near an ADE singularity and via holography to the topological

[12]Earlier indications of such a relation can be found in [18], [48] (see also [32]).

sector of little string theory [13–15].

(2) The double-scaled ADE matrix models of [33].

(3) The generalized Kontsevich matrix models [49, 50].

(4) The $\mathcal{N}=2$ topologically twisted coset $SU(2)_k/U(1)$ coupled to topological gravity [51, 52, 47, 53, 46, 54].

(5) The B-model $\mathcal{N}=2$ topological string on a hypersurface of the form [50]

$$z_1^k + z_2 + z_3^2 + z_4^2 = 0 \ . \tag{4.1}$$

(6) The B-model $\mathcal{N}=2$ topological string on $H_3^+ \times S^3$ [55].

A better understanding of the structure of the minimal $\mathcal{N}=4$ topological string both in the closed string and open string sector would determine whether (1) is a bona fide member of this set of theories.

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A. The $\mathcal{N}=4$ superconformal algebra

The (small) $\mathcal{N}=4$ supeconformal algebra (SCA) comprises of the $\mathcal{N}=2$ superconformal algebra generators $(T, G^{\pm}, J)$ plus two additional currents of charge ± 2, which will be denoted as J^{++} and J^{--}.[13] The three currents (J, J^{++}, J^{--}) form an $SU(2)$ algebra, under which the $G^{\pm}$ currents can be used to generate two more fermionic generators $\tilde{G}^{\pm}$. Altogether these four fermionic generators form the $SU(2)$ doublets

$$(G^+, \tilde{G}^-) \ , \quad (\tilde{G}^+, G^-) \ . \tag{A.1}$$

In unitary theories the adjoint is defined by

$$G^{+\dagger} = G^- \ , \quad \tilde{G}^{+\dagger} = \tilde{G}^- \ , \quad J^{++\dagger} = J^{--} \ . \tag{A.2}$$

The $\mathcal{N}=4$ SCA OPE's are

$$J^{--}(z)J^{++}(0) \sim \frac{\hat{c}}{2z^2} - \frac{J(0)}{z} \ , \tag{A.3}$$

$$J^{--}(z)G^{+}(0) \sim \frac{\tilde{G}^-(0)}{z} \ , \quad J^{++}(z)\tilde{G}^{-}(0) \sim -\frac{G^+(0)}{z} \ , \tag{A.4}$$

$$J^{++}(z)G^{-}(0) \sim \frac{\tilde{G}^+(0)}{z} \ , \quad J^{--}(z)\tilde{G}^{+}(0) \sim -\frac{G^-(0)}{z} \ , \tag{A.5}$$

[13] We are following the notation of [7].

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$$J(z)G^{\pm}(0) \sim \pm\frac{G^{\pm}(0)}{z}\ , \quad J(z)\tilde{G}^{\pm}(0) \sim \pm\frac{\tilde{G}^{\pm}(0)}{z}\ , \tag{A.6}$$

$$J^{--}\cdot G^{-} \sim J^{++}\cdot G^{+} \sim J^{++}\cdot \tilde{G}^{+} \sim J^{--}\cdot \tilde{G}^{-} \sim 0\ , \tag{A.7}$$

$$G^{\pm}\cdot G^{\pm} \sim \tilde{G}^{\pm}\cdot \tilde{G}^{\pm} \sim G^{+}\cdot \tilde{G}^{-} \sim G^{-}\cdot \tilde{G}^{+} \sim 0\ , \tag{A.8}$$

$$G^{+}(z)G^{-}(0) \sim \frac{\hat{c}}{z^3} + \frac{J(0)}{z^2} + \frac{2T(0)+\partial J(0)}{z}\ , \tag{A.9}$$

$$\tilde{G}^{+}(z)\tilde{G}^{-}(0) \sim \frac{\hat{c}}{z^3} + \frac{J(0)}{z^2} + \frac{2T(0)+\partial J(0)}{z}\ , \tag{A.10}$$

$$G^{+}(z)\tilde{G}^{+}(0) \sim \frac{J^{++}(0)}{z^2} + \frac{\partial J^{++}(0)}{2z}\ , \tag{A.11}$$

$$G^{-}(z)\tilde{G}^{-}(0) \sim \frac{J^{--}(0)}{z^2} + \frac{\partial J^{--}(0)}{2z}\ . \tag{A.12}$$

Any theory with $\hat{c} = 2$, $\mathcal{N} = 2$ superconformal symmetry and integral $U(1)$ charges automatically has $\mathcal{N} = 4$ superconformal symmetry with the extra generators provided by the spectral flow operators $J^{\pm\pm} = e^{\pm\int J}$.

B. $SL(2,\mathbb{R})$ and $SU(2)$ conventions

In this appendix, we summarize our conventions for the $SL(2)$ and $SU(2)$ WZW models. For $SL(2)$ we follow the conventions of [20], where

$$\Phi^{(sl)}_{j,m,\bar{m}} = \int d^2x\ x^{-j-1+m}\bar{x}^{-j-1+\bar{m}}\Phi^{(sl)}_{j}(x,\bar{x}) \tag{B.1}$$

and

$$J^3(z)\Phi^{(sl)}_{j,m}(0) \sim \frac{m}{z}\Phi^{(sl)}_{j,m}(0)\ , \quad J^{\pm}(z)\Phi^{(sl)}_{j,m}(0) \sim \frac{m\pm(j+1)}{z}\Phi^{(sl)}_{j,m\pm 1}(0)\ . \tag{B.2}$$

The semiclassical behavior of $\Phi^{(sl)}(x)$ in the $(\gamma,\bar{\gamma},\phi)$ coordinate system is [20]

$$\Phi^{(sl)}_{j,cl}(x) = \frac{2j+1}{\pi}\Big(|\gamma-x|^2e^{\phi}+e^{-\phi}\Big)^{2j}\ . \tag{B.3}$$

At the conformal boundary of $SL(2)$ $\phi\to\infty$ and $\Phi^{(sl)}_{j,cl}$ can be expanded as (see *e.g.* eq. (2.8) in [56])

$$\Phi^{(sl)}_{j,cl} \simeq -e^{-2(j+1)\phi}\delta^{(2)}(\gamma-x)+\mathcal{O}(e^{-2(j+2)\phi}) - \frac{(2j+1)|\gamma-x|^{4j}}{\pi}e^{2j\phi}+\mathcal{O}(e^{2(j-1)\phi})\ . \tag{B.4}$$

As a result, for $j>0$, $\Phi^{(sl)}_{j,m,\bar{m}}$ will give rise to a normalizable vertex operator in (2.1) and $\Phi^{(sl)}_{-j-1,m,\bar{m}}$ will give rise to non-normalizable one. The two vertex operators are related by the reflection relation

$$\Phi^{(sl)}_{j,m,\bar{m}} = R(j,m,\bar{m})\Phi^{(sl)}_{-j-1,m,\bar{m}}\ , \tag{B.5}$$

where

$$R(j,m,\bar{m}) = k\left[\frac{1}{k\pi}\gamma\left(\frac{1}{k}\right)\right]^{-2j-1}\gamma\left(1+\frac{2j+1}{k}\right)\frac{\Gamma(2j+1)\Gamma(-j-m)\Gamma(-j+\bar{m})}{\Gamma(-2j)\Gamma(1+j-m)\Gamma(1+j+\bar{m})} \tag{B.6}$$

denotes the $SL(2,\mathbb{R})$ reflection coefficient.

In $SU(2)$ the OPE's of the currents with the primary fields $\Phi^{(su)}_{j,m}$ are

$$K^3(z)\Phi^{(su)}_{j,m}(0) \sim \frac{m}{z}\Phi^{(su)}_{j,m}(0) \ , \quad K^{\pm}(z)\Phi^{(su)}_{j,m}(0) \sim \frac{j \mp m}{z}\Phi^{(su)}_{j,m\pm1}(0) \ . \tag{B.7}$$

For quick reference, we also summarize in this appendix our conventions for the spectral flow operation in the $\mathcal{N}=2$ superconformal algebra, the bosonic $SL(2)_{k+2}$ WZW model and the bosonic $SU(2)_{k-2}$ WZW model.

For the $\mathcal{N}=2$ superconformal algebra with central charge c the spectral flow transformation by an amount θ is an automorphism of the form

$$\tilde{L}_n = L_n + \theta J_n + \theta^2 \frac{c}{6}\delta_{n,0} \ , \quad \tilde{G}^{\pm}_n = G^{\pm}_{n\pm\theta} \ , \quad \tilde{J}_n = J_n + \frac{c}{3}\theta \ . \tag{B.8}$$

On the level of primary fields this transformation can be achieved by multiplying with the vertex operator $e^{-i\theta\sqrt{\frac{c}{3}}X_R}$, where X_R is a canonically normalized boson that bosonizes the $U(1)_R$ current.

For the $SL(2)_{k+2}$ WZW model the spectral flow operation with winding number $w \in \mathbb{Z}$ is defined by the current automorphism

$$\tilde{L}_n = L_n - wJ^3_n - \frac{k+2}{4}w^2\delta_{n,0} \ , \quad \tilde{J}^3_n = J^3_n - \frac{k+2}{2}w\delta_{n,0} \ , \quad \tilde{J}^{\pm}_n = J^{\pm}_{n\mp w} \ . \tag{B.9}$$

On vertex operators it corresponds to multiplication with the operator $e^{w\sqrt{\frac{k+2}{2}}X^{(sl)}}$. $X^{(sl)}$ is the canonically normalized boson that bosonizes the current J^3, *i.e.*

$$J^3 = -\sqrt{\frac{k+2}{2}}\partial X^{(sl)} \ . \tag{B.10}$$

In a similar fashion, for the $SU(2)_{k-2}$ WZW model the spectral flow operation with winding number $w \in \mathbb{Z}$ is defined by the current automorphism

$$\tilde{L}_n = L_n + wK^3_n + \frac{k-2}{4}w^2\delta_{n,0} \ , \quad \tilde{K}^3_n = K^3_n - \frac{k-2}{2}w\delta_{n,0} \ , \quad \tilde{K}^{\pm}_n = K^{\pm}_{n\pm w} \ . \tag{B.11}$$

On vertex operators it corresponds to multiplication with the operator $e^{iw\sqrt{\frac{k-2}{2}}X^{(su)}}$. $X^{(su)}$ is the canonically normalized boson that bosonizes the current K^3, *i.e.*

$$K^3 = i\sqrt{\frac{k-2}{2}}X^{(su)} \ . \tag{B.12}$$

The spectral flowed primary vertex operators $\Phi^{(sl)}_{j,m}$, $\Phi^{(su)}_{j,m}$ will be denoted as $\Phi^{(sl),w}_{j,m}$, $\Phi^{(su),w}_{j,m}$.

In the main text we obtain the $\frac{SL(2)_k}{U(1)} \times \frac{SU(2)_k}{U(1)}$ theory by gauging the null $U(1)$ current

$$J_g = i\partial H_2 - i\partial H_1 + K_3 - J_3 \tag{B.13}$$

in $SL(2)_k \times SU(2)_k$. As in [29], [30] it will be useful to define the gauge equivalent vertex operators

$$\mathcal{O} \sim \mathcal{O}\ e^{\pm\int J_g} = \mathcal{O}\ e^{\pm(iH_2 - iH_1 + i\sqrt{\frac{k-2}{2}}X^{su} + \sqrt{\frac{k+2}{2}}X^{sl})} \ . \tag{B.14}$$

In particular, for $\mathcal{O} = \mathcal{O}^+_{2j+1}$ we get

$$\mathcal{O}^+_{2j+1} \sim e^{-\int J_g}\ \mathcal{O}^+_{2j+1} = e^{-iH_2}\Phi^{(sl),w=-1}_{j,j+1}\Phi^{(su),w=-1}_{j,j}\ . \tag{B.15}$$

Using the $SL(2)$ representation equivalence $\mathcal{D}^-_{-j-\frac{k+2}{2}} = \mathcal{D}^{+,w=-1}_j$, *i.e.*

$$\Phi^{(sl)}_{j+\frac{k}{2},-j-\frac{k+2}{2}} = \Phi^{(sl),w=-1}_{-j-1,-j} \tag{B.16}$$

and the corresponding $SU(2)$ identity

$$\Phi^{(su),w=-1}_{j,j} = \Phi^{(su)}_{\frac{k-2}{2}-j,j-\frac{k-2}{2}} \tag{B.17}$$

we find

$$\mathcal{O}^+_{2j+1} = e^{-iH_2}\Phi^{(sl)}_{\frac{k-2}{2}-j,j-\frac{k}{2}}\Phi^{(su)}_{\frac{k-2}{2}-j,j-\frac{k-2}{2}} = \mathcal{O}^-_{k-2j-1} \tag{B.18}$$

thus reproducing in the $\mathcal{N}=4$ topological string the $\mathcal{N}=2$ string relation that appears in eq. (5.34) of [12]. For reference, we also list the representation identities

$$\Phi^{(sl),w=1}_{-j-1,j} = \Phi^{(sl)}_{j+\frac{k}{2},j+\frac{k+2}{2}}\ ,\quad \Phi^{(su),w=1}_{j,-j} = \Phi^{(su)}_{\frac{k-2}{2}-j,\frac{k-2}{2}-j}\ . \tag{B.19}$$

C. Summary of the Stoyanovsky-Ribault-Teschner map

Recently the authors of [20, 22] formulated a precise map between sphere correlation functions in the $SL(2,\mathbb{C})/SU(2)$ WZW model and correlation functions in Liouville field theory. The basic formula that was obtained in [20] provides a map between winding number conserving N-point functions in the $SL(2,\mathbb{C})/SU(2)$ WZW model at level k and $(2N-2)$-point functions in Liouville field theory with linear dilaton slope $Q = b + b^{-1}$ and $b^2 = \frac{1}{k-2}$. In this paper, we are interested, as in [30], in winding number violating amplitudes. For such amplitudes the relevant formula is [22][14]

$$\begin{aligned}\left\langle \prod_{i=1}^N \Phi^{(sl),w_i}_{j_i,m_i,\bar m_i}(z_i,\bar z_i)\right\rangle_{\sum w_i = r\geq 0} &= \prod_{i=1}^N \mathcal{N}^{j_i}_{m_i,\bar m_i}\delta^{(2)}\left(\sum_{\ell=1}^N m_\ell + \frac{k}{2}r\right)\times \\ \prod_{a=1}^{N-2-r}\int d^2y_a\ \tilde{\mathcal{F}}_k(z_i,\bar z_i;y_a,\bar y_a)&\left\langle \mathcal{V}_{\alpha_i}(z_i,\bar z_i)\mathcal{V}_{-\frac{1}{2b}}(y_a,\bar y_a)\right\rangle\ ,\end{aligned} \tag{C.1}$$

where for c_k a k-dependent constant,

$$\begin{aligned}\tilde{\mathcal{F}}_k(z_i,\bar z_i;y_a,\bar y_a) = \frac{2\pi^{3-2N}bc_k^r}{(N-2-r)!}&\prod_{i<j\leq N}(z_i-z_j)^{\beta_{ij}}(\bar z_i-\bar z_j)^{\bar\beta_{ij}} \\ &\prod_{a<b\leq N-2-r}|y_a-y_b|^k\prod_{r=1}^N\prod_{c=1}^{N-2-r}(z_r-y_c)^{-m_r-\frac{k}{2}}(\bar z_r-\bar y_c)^{-\bar m_r-\frac{k}{2}}\ ,\end{aligned} \tag{C.2}$$

[14]Compared to the conventions of [22] we have $m_{ours} = -m_{there}$, $\bar m_{ours} = -\bar m_{there}$ and $w_{ours} = -w_{there}$.

with

$$\begin{aligned}\beta_{ij} &= \frac{k}{2} + m_i + m_j - \frac{k}{2}w_i w_j - w_i m_j - w_j m_i \ , \\ \bar\beta_{ij} &= \frac{k}{2} + \bar m_i + \bar m_j - \frac{k}{2}\bar w_i \bar w_j - \bar w_i \bar m_j - \bar w_j \bar m_i\end{aligned} \tag{C.3}$$

and

$$\mathcal{N}^j_{m,\bar m} = \frac{\Gamma(-j+m)}{\Gamma(1+j-\bar m)} \ , \tag{C.4}$$

$$\alpha_i = bj_i + b + \frac{1}{2b} \ . \tag{C.5}$$

$\mathcal{V}_a$ are vertex operators in Liouville field theory with asymptotic (weak coupling) form

$$\mathcal{V}_a = e^{\sqrt{2}a\phi} \ . \tag{C.6}$$

In our conventions $\alpha' = 2$ and the Liouville interaction takes the form $\mu_L e^{\sqrt{2}b\phi}$. The Liouville field theory central charge is

$$c_L = 1 + 6Q^2 \ . \tag{C.7}$$

The scaling dimension of $\mathcal{V}_a$ is

$$\Delta(\mathcal{V}_a) = a(Q_L - a) \ . \tag{C.8}$$

It will be useful to consider the analytic continuation of the above expressions to $SU(2)_k$. This entails taking $k \to -k$ and setting $j \to -j-1$, $m \to -m, \bar m \to -\bar m$. Then we find that we can recast a correlation function of $SU(2)_k$ operators in terms of a (Coulomb-Gas) correlation function in a linear dilaton theory with imaginary slope $\tilde b + \tilde b^{-1}$, where $\tilde b^2 = -\frac{1}{k+2}$. The analytically continued version of (C.1) reads

$$\begin{aligned}&\left\langle \prod_{i=1}^N \Phi^{(su),w_i}_{j_i,m_i,\bar m_i}(z_i,\bar z_i)\right\rangle_{\sum w_i = r\geq 0} = \prod_{i=1}^N \tilde{\mathcal{N}}^{j_i}_{m_i,\bar m_i}\delta^{(2)}\left(\sum_{\ell=1}^N m_\ell + \frac{k}{2}r\right) \times \\ &\prod_{a=1}^{N-2-r}\int d^2y_a\ \tilde{\tilde{\mathcal{F}}}_k(z_i,\bar z_i;y_a,\bar y_a)\left\langle \tilde{\mathcal{V}}_{\alpha_i}(z_i,\bar z_i)\tilde{\mathcal{V}}_{-\frac{1}{2\tilde b}}(y_a,\bar y_a)\right\rangle \ ,\end{aligned} \tag{C.9}$$

where for $\tilde c_k$ a k-dependent constant,

$$\begin{aligned}\tilde{\tilde{\mathcal{F}}}_k(z_i,\bar z_i;y_a,\bar y_a) &= \frac{2\pi^{3-2N}\tilde b \tilde c_k^r}{(N-2-r)!}\prod_{i<j\leq N}(z_i-z_j)^{-\beta_{ij}}(\bar z_i - \bar z_j)^{-\bar\beta_{ij}} \\ &\prod_{a<b\leq N-2-r}|y_a-y_b|^{-k}\prod_{r=1}^N\prod_{c=1}^{N-2-r}(z_r-y_c)^{m_r+\frac{k}{2}}(\bar z_r - \bar y_c)^{\bar m_r + \frac{k}{2}} \ ,\end{aligned} \tag{C.10}$$

with β_{ij} and $\bar\beta_{ij}$ as in (C.3) and

$$\tilde{\mathcal{N}}^j_{m,\bar m} = \frac{\Gamma(1+j-m)}{\Gamma(-j+\bar m)} \ , \tag{C.11}$$

$$\alpha = -\tilde b j + \frac{1}{2\tilde b} \ . \tag{C.12}$$

$\tilde{\mathcal{V}}_a$ are vertex operators of the form $e^{\sqrt{2}a\tilde\phi}$ in the Coulomb-Gas representation of the $(1, k+2)$ minimal model.

D. The $(1,k)$ minimal bosonic string

The $(1,k)$ minimal bosonic string is obtained by coupling the $(1,k)$ minimal model with Liouville field theory in such a way that the total central charge is 26. Despite the fact that the $(1,k)$ minimal models are outside the range of definition of the "minimal" Virasoro series the $(1,k)$ minimal bosonic string is well-defined [54]. The minimal model has central charge

$$c = 1 - 6\frac{(k-1)^2}{k} \ . \tag{D.1}$$

This implies that the linear dilaton slope of Liouville theory is $Q = \frac{1}{\sqrt{k}} + \sqrt{k}$.

The $(1,k)$ minimal string has an infinite number of physical states $T_{r,s}$, which are labeled by two integers r, s [57, 55]. The corresponding vertex operators are

$$T_{r,s} = cW_{r,s}e^{\sqrt{2}\left(\frac{r+1}{2\sqrt{k}}+\frac{s}{2}\sqrt{k}\right)\phi} \ , \quad r = 1,2,...,k-1 \ , \ s = 1,2,3,... \ , \tag{D.2}$$

where ϕ is the Liouville field theory boson and $W_{r,s}$ are states of the $(1,k)$ minimal model. In the Coulomb-Gas representation the latter take the form

$$W_{r,s} = e^{i\sqrt{2}\left(\frac{r-1}{2\sqrt{k}}+\frac{s}{2}\sqrt{k}\right)\tilde{\phi}} \ . \tag{D.3}$$

$\tilde{\phi}$ is a free boson with linear dilaton slope $\tilde{Q} = \frac{1-k}{\sqrt{k}}$. In our conventions, the vertex operator $\tilde{\mathcal{V}}_\alpha = e^{i\sqrt{2}\alpha\tilde{\phi}}$ has scaling dimension

$$\Delta(\tilde{\mathcal{V}}_\alpha) = \alpha(\tilde{Q} + \alpha) \ . \tag{D.4}$$

In the main text, the most important role will be played by the physical vertex operators $T_{r,1}$, which we denote as

$$\mathcal{T}_r = T_{r,1} = c\bar{c}\ \mathcal{V}_{\frac{r+1}{2\sqrt{k}}+\frac{\sqrt{k}}{2}}\tilde{\mathcal{V}}_{\frac{r-1}{2\sqrt{k}}+\frac{\sqrt{k}}{2}} \ . \tag{D.5}$$

By definition $\mathcal{V}_\alpha = e^{\sqrt{2}\alpha\phi}$.

References

[1] A. Neitzke and C. Vafa, *Topological strings and their physical applications*, hep-th/0410178.

[2] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255].

[3] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[4] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [hep-th/9307158].

[5] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

JHEP03(2006)045

[6] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[7] N. Berkovits and C. Vafa, $N = 4$ *topological strings*, *Nucl. Phys.* **B 433** (1995) 123 [hep-th/9407190].

[8] M. Ademollo et al., *Dual string with U(1) color symmetry*, *Nucl. Phys.* **B 111** (1976) 77.

[9] H. Ooguri and C. Vafa, *Selfduality and* $N = 2$ *string magic*, *Mod. Phys. Lett.* **A 5** (1990) 1389.

[10] H. Ooguri and C. Vafa, *Geometry of* $N = 2$ *strings*, *Nucl. Phys.* **B 361** (1991) 469.

[11] N. Marcus, *A tour through* $N = 2$ *strings*, hep-th/9211059.

[12] O. Aharony, B. Fiol, D. Kutasov and D.A. Sahakyan, *Little string theory and heterotic/type-II duality*, *Nucl. Phys.* **B 679** (2004) 3 [hep-th/0310197].

[13] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, *Nucl. Phys.* **B 463** (1996) 55 [hep-th/9511164].

[14] A. Giveon, D. Kutasov and O. Pelc, *Holography for non-critical superstrings*, *JHEP* **10** (1999) 035 [hep-th/9907178].

[15] A. Giveon and D. Kutasov, *Little string theory in a double scaling limit*, *JHEP* **10** (1999) 034 [hep-th/9909110].

[16] O. Aharony, *A brief review of 'little string theories'*, *Class. and Quant. Grav.* **17** (2000) 929 [hep-th/9911147].

[17] D. Kutasov, *Introduction to little string theory*, prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 2-10 Apr 2001.

[18] A. Konechny, A. Parnachev and D.A. Sahakyan, *The ground ring of* $N = 2$ *minimal string theory*, *Nucl. Phys.* **B 729** (2005) 419 [hep-th/0507002].

[19] A.V. Stoyanovsky, *A relation between the Knizhnik-Zamolodchikov and Belavin-Polyakov-Zamolodchikov systems of partial differential equations*, math.ph/0012013.

[20] S. Ribault and J. Teschner, $H(3)^+$ *correlators from Liouville theory*, *JHEP* **06** (2005) 014 [hep-th/0502048].

[21] G. Giribet and Y. Nakayama, *The Stoyanovsky-Ribault-teschner map and string scattering amplitudes*, hep-th/0505203.

[22] S. Ribault, *Knizhnik-zamolodchikov equations and spectral flow in* AdS_3 *string theory*, *JHEP* **09** (2005) 045 [hep-th/0507114].

[23] C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers and P. Vanhove, *Heterotic/type-I duality and D-brane instantons*, *Nucl. Phys.* **B 509** (1998) 33 [hep-th/9707126].

[24] E. Kiritsis, *Duality and instantons in string theory*, hep-th/9906018.

[25] S. Stieberger and T.R. Taylor, *Non-abelian Born-Infeld action and type-I - heterotic duality, II. Nonrenormalization theorems*, *Nucl. Phys.* **B 648** (2003) 3 [hep-th/0209064].

[26] S. Mukhi and C. Vafa, *Two-dimensional black hole as a topological coset model of* $c = 1$ *string theory*, *Nucl. Phys.* **B 407** (1993) 667 [hep-th/9301083].

[27] N. Ohta and H. Suzuki, *Bosonization of a topological coset model and noncritical string theory*, *Mod. Phys. Lett.* **A 9** (1994) 541 [hep-th/9310180].

[28] D. Ghoshal and C. Vafa, $c = 1$ *string as the topological theory of the conifold*, *Nucl. Phys.* **B 453** (1995) 121 [hep-th/9506122].

[29] T. Takayanagi, $c < 1$ *string from two dimensional black holes*, *JHEP* **07** (2005) 050 [hep-th/0503237].

[30] S. Nakamura and V. Niarchos, *Notes on the S-matrix of bosonic and topological non- critical strings*, *JHEP* **10** (2005) 025 [hep-th/0507252].

[31] S.K. Ashok, S. Murthy and J. Troost, *Topological cigar and the* $c = 1$ *string: open and closed*, *JHEP* **02** (2006) 013 [hep-th/0511239].

[32] L. Rastelli, *Minimal superstrings, topological strings and NSF5 branes*, talk given at the Simons workshop in Mathematics and Physics 2005, the talk can be found online at http://insti.physics.sunysb.edu/itp/conf/simonswork3/.

[33] I.K. Kostov, *Gauge invariant matrix model for the A-D-E closed strings*, *Phys. Lett.* **B 297** (1992) 74 [hep-th/9208053].

[34] D.A. Sahakyan and T. Takayanagi, *On the connection between* $N = 2$ *minimal string and (1,n) bosonic minimal string*, hep-th/0512112.

[35] E. Kiritsis, C. Kounnas and D. Lüst, *A large class of new gravitational and axionic backgrounds for four-dimensional superstrings*, *Int. J. Mod. Phys.* **A 9** (1994) 1361 [hep-th/9308124].

[36] D. Israel, C. Kounnas, A. Pakman and J. Troost, *The partition function of the supersymmetric two-dimensional black hole and little string theory*, *JHEP* **06** (2004) 033 [hep-th/0403237].

[37] N. Itzhaki, D. Kutasov and N. Seiberg, *I-brane dynamics*, *JHEP* **01** (2006) 119 [hep-th/0508025].

[38] W. Lerche, C. Vafa and N.P. Warner, *Chiral rings in* $N = 2$ *superconformal theories*, *Nucl. Phys.* **B 324** (1989) 427.

[39] Y.-K.E. Cheung, Y. Oz and Z. Yin, *Families of* $N = 2$ *strings*, *JHEP* **11** (2003) 026 [hep-th/0211147].

[40] O. Aharony, A. Giveon and D. Kutasov, *LSZ in LST*, *Nucl. Phys.* **B 691** (2004) 3 [hep-th/0404016].

[41] E. Witten, *Mirror manifolds and topological field theory*, hep-th/9112056.

[42] N.P. Warner, *N=2 supersymmetric integrable models and topological field theories*, hep-th/9301088.

[43] A.B. Zamolodchikov and V.A. Fateev, *Operator algebra and correlation functions in the two-dimensional Wess-Zumino* $\mathrm{SU}(2) \times \mathrm{SU}(2)$ *chiral model*, *Sov. J. Nucl. Phys.* **43** (1986) 657.

[44] J. Teschner, *On structure constants and fusion rules in the* $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$ *WZNW model*, *Nucl. Phys.* **B 546** (1999) 390 [hep-th/9712256].

[45] J. Teschner, *Operator product expansion and factorization in the H-3+ WZNW model*, *Nucl. Phys.* **B 571** (2000) 555 [hep-th/9906215].

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[46] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B 371** (1992) 191.

[47] R. Dijkgraaf, H.L. Verlinde and E.P. Verlinde, *Topological strings in $D < 1$*, *Nucl. Phys.* **B 352** (1991) 59.

[48] T. Takayanagi, *Notes on s-matrix of non-critical $N = 2$ string*, *JHEP* **09** (2005) 001 [hep-th/0507065].

[49] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[50] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino and C. Vafa, *Topological strings and integrable hierarchies*, *Commun. Math. Phys.* **261** (2006) 451 [hep-th/0312085].

[51] T. Eguchi and S.-K. Yang, *N=2 superconformal models as topological field theories*, *Mod. Phys. Lett.* **A 5** (1990) 1693.

[52] R. Dijkgraaf and E. Witten, *Mean field theory, topological field theory and multimatrix models*, *Nucl. Phys.* **B 342** (1990) 486.

[53] K. Li, *Topological gravity with minimal matter*, *Nucl. Phys.* **B 354** (1991) 711.

[54] R. Dijkgraaf, *Intersection theory, integrable hierarchies and topological field theory*, hep-th/9201003.

[55] L. Rastelli and M. Wijnholt, *Minimal AdS_3*, hep-th/0507037.

[56] D. Kutasov and N. Seiberg, *More comments on string theory on AdS_3*, *JHEP* **04** (1999) 008 [hep-th/9903219].

[57] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, *Extended $N = 2$ superconformal structure of gravity and W gravity coupled to matter*, *Nucl. Phys.* **B 401** (1993) 304 [hep-th/9211040].